Gaussian random eigenfunctions and spatial correlations in quantum dots

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We show that spatial correlations in wave functions of quantum dots, obtained earlier by averaging over a random potential via supermatrix techniques, can be computed with much less effort by making use of Berry's conjecture that the energy eigenfunctions in a quantized chaotic system are Gaussian random variables. Furthermore, in the case of a time-reversal invariant system, we find a greatly simplified (though equivalent) formula for these correlations. [S1063-651X(96)09505-0]

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Prigodin [1] and Prigodin *et al.* [2] have computed the joint probability distribution

$$P(v_1,v_2)=\langle \delta(v_1-V|\psi(\mathbf{x})|^2)\delta(v_2-V|\psi(\mathbf{y})|^2)\rangle \tag{1}$$

for the squared amplitude of an energy eigenfunction $\psi(\mathbf{x})$ at two different points ($\mathbf{x}$ and $\mathbf{y}$) in a quantum dot with volume V, assuming either broken [1] or unbroken [2] time-reversal invariance. This was accomplished by averaging over a random potential using supermatrix techniques. Here we show that their results can be obtained much more easily by making use of Berry's conjecture [3] that the energy eigenfunctions in a quantized chaotic system are Gaussian random variables. Furthermore, in the time-reversal invariant case, we get a formula for $P(v_1,v_2)$ which is considerably simpler than (though mathematically equivalent to) the one given in [2].

We interpret Berry's conjecture as implying that

$$P(\psi)\propto\exp\left[-\frac{\beta}{2}\int d\mathbf{x}d\mathbf{y}\,\psi^*(\mathbf{x})K(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\right], \tag{2}$$

where $P(\psi)$ is the probability that a particular energy eigenfunction (with a definite energy eigenvalue) is equal to the specified function $\psi(\mathbf{x})$. Here $\beta=1$ for a system which is time-reversal invariant, and $\beta=2$ for a system which is not. In the former case, $\psi(\mathbf{x})$ is a real function. In either case, the kernel $K(\mathbf{x},\mathbf{y})$ is the inverse of the two-point correlation function $\langle\psi^*(\mathbf{x})\psi(\mathbf{y})\rangle=V^{-1}f(|\mathbf{x}-\mathbf{y}|)$, where the angular brackets now denote an average over $P(\psi)$, rather than over a random potential. We note that an assumption equivalent to Eq. (2) was made in [4] to calculate the probability distribution of level widths and conductance peaks in a quantum dot with attached leads. We will not need the explicit formula for $f(r)$ [3], but notice that proper normalization of the wave function requires $f(0)=1$.

To get $P(v_1,v_2)$ from Eq. (2), we note that integrating out all variables except $\psi_1=\psi(\mathbf{x})$ and $\psi_2=\psi(\mathbf{y})$ will yield a Gaussian in these variables, and this Gaussian must reproduce the correct two-point correlation functions. Thus we conclude that

$$P(\psi_1,\psi_2)\propto(\det M)^{-\beta/2}\exp\left[-\frac{\beta}{2}\psi_i^*(M^{-1})_{ij}\psi_j\right], \tag{3}$$

where $M_{ij}=\langle\psi_i^*\psi_j\rangle=V^{-1}[\delta_{ij}+(1-\delta_{ij})f]$, and $f=f(|\mathbf{x}-\mathbf{y}|)$.

For the time-reversal invariant case, $\beta=2$ and ψ_i is complex; changing integration variables to $v_i=V|\psi_i|^2$ and $\theta_i=\arg\psi_i$, including a proper Jacobian, and integrating over θ_1 and θ_2 yields

$$P(v_1,v_2)=\frac{1}{1-f^2}\exp\left(-\frac{v_1+v_2}{1-f^2}\right)I_0\left(\frac{2f\sqrt{v_1v_2}}{1-f^2}\right), \tag{4}$$

where $I_0(z)$ is a modified Bessel function, and $f=f(|\mathbf{x}-\mathbf{y}|)$. This is the same as Eq. (15) of [1].

For the time-reversal invariant case, $\beta=1$ and ψ_i is real; changing integration variables to $v_i=V\psi_i^2$ and including a proper Jacobian then yields

$$P(v_1,v_2)=\frac{1}{2\pi(1-f^2)^{1/2}(v_1v_2)^{1/2}}\exp\left(-\frac{v_1+v_2}{2(1-f^2)}\right)\times\cosh\left(\frac{f\sqrt{v_1v_2}}{1-f^2}\right). \tag{5}$$

This should be compared with Eq. (5) of [2], in which $P(v_1,v_2)$ is expressed as a parametric double integral. To verify that these two very different expressions for $P(v_1,v_2)$ are equivalent, we compute the moments

$$Q_{nm}=\int_0^\infty dv_1dv_2v_1^nv_2^mP(v_1,v_2)=V^{n+m}\langle|\psi_1|^{2n}|\psi_2|^{2m}\rangle. \tag{6}$$

Using standard combinatoric properties of Gaussian distributions, we have, in the time-reversal invariant case when $\psi(\mathbf{x})$ is real,

$$\langle\psi_1\cdots\psi_{2p}\rangle=\sum_{\text{pairs}}\langle\psi_{i_1}\psi_{i_2}\rangle\cdots\langle\psi_{i_{2p-1}}\psi_{i_{2p}}\rangle, \tag{7}$$

where the sum is over the $(2p-1)!!$ ways of pairing up all the ψ's. Recalling that $\langle\psi_1\psi_1\rangle=\langle\psi_2\psi_2\rangle=1/V$ and $\langle\psi_1\psi_2\rangle=f/V$, the last line of Eq. (6) is easily evaluated as a special case of Eq. (7). To find the contribution to Q_{nm} which is proportional to f^{2q}, where q is an integer, we choose $2q$ of the $2n$ ψ_1's in $(2n)!/(2q)!(2n-2q)!$ ways,

$2q$ of the $2m$ ψ_2's in $(2m)!/(2q)!(2m-2q)!$ ways, and pair them up in $(2q)!$ ways. Then we pair up the remaining $2n-2q$ ψ_1's with each other in $(2n-2q-1)!!$ ways, and the remaining $2m-2q$ ψ_2's with each other in $(2m-2q-1)!!$ ways. Putting all of this together, and using the identity $(2p-1)!!=(2p)!/2^p p!$, we find

$$Q_{nm}=\sum_{q=0}^{\min(n,m)}\frac{(2n)!(2m)!f^{2q}}{2^{n+m-2q}(n-q)!(m-q)!(2q)!}. \tag{8}$$

This is equivalent to Eq. (21) of [2] (as corrected in the erratum).

For completeness, we note that we can also easily compute Q_{nm} in the case of broken time-reversal invariance. The analog of Eq. (7) is

$$\langle\psi_1^*\psi_1\cdots\psi_p^*\psi_p\rangle=\sum_{\text{perms}}\langle\psi_1^*\psi_{i_1}\rangle\cdots\langle\psi_p^*\psi_{i_p}\rangle, \tag{9}$$

where the sum is over the $p!$ permutations of the ψ_i's. We now have $\langle\psi_1^*\psi_1\rangle=\langle\psi_2^*\psi_2\rangle=1/V$ and $\langle\psi_1^*\psi_2\rangle=\langle\psi_2^*\psi_1\rangle=f/V$. To find the contribution to Q_{nm} which is proportional to f^{2q}, we choose q of the n ψ_1's in $n!/q!(n-q)!$ ways, q of the m ψ_2^*'s in $m!/q!(m-q)!$ ways, and pair them up in $q!$ ways; we also choose q of the n ψ_1^*'s in $n!/q!(n-q)!$ ways, q of the m ψ_2's in $m!/q!(m-q)!$ ways, and pair them up in $q!$ ways. Then we pair the remaining $n-q$ ψ_1's and $n-q$ ψ_1^*'s with each other in $(n-q)!$ ways, and the remaining $m-q$ ψ_2's and $m-q$ ψ_2^*'s with each other in $(m-q)!$ ways. Putting all of this together, we find

$$Q_{nm}=\sum_{q=0}^{\min(n,m)}\frac{(n!)^2(m!)^2f^{2q}}{(n-q)!(m-q)!(q!)^2}. \tag{10}$$

This same formula follows from Eq. (14) of [1] (after expanding in powers of f and doing the contour integral in that equation term by term).

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